"AVERAGING" METHOD IN PROBLEMS OF CONVECTIVE MASS AND HEAT TRANSFER

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UDC 532.7

An approximate method is proposed for the solution of nonlinear boundary problems of convective heat and mass transfer; the method is based on the procedure of "averaging" equations or boundary conditions.

The complexity of mathematical problems of chemical technology, as a rule, excludes the possibility of obtaining accurate analytical solutions. However, there often arises a situation in which the basic interest is not in accurate solutions themselves (they may be cumbersome and inexpedient for practical use) but in approximate analytical dependences, which correctly reflect the qualitative influence of the basic parameters on the given process. This is primarily because measurements in real processes may never be accurately reproduced. In the present work, two approximate methods (based on the idea of averaging) are proposed for solving boundary problems of convective mass and heat transfer; these methods may be used in engineering practice.

1. Method of Linearization of Averaging

In investigating problems of chemical mechanics with boundary conditions of the first kind (it is assumed that the desired function c takes a constant value at the boundaries), a simple approximate method based on the replacement of any nonlinear monotonic function $\Phi = \Phi(c)$ by the linear function Φ^* often proves useful

$$\Phi(c) \to \Phi^*(c) = A + Bc, \tag{1}$$

where the constant coefficients A and B are determined from the conditions of integral equality of Φ and Φ^* on average

$$A + \frac{1}{2}B = \langle \Phi \rangle, \quad \langle \Phi \rangle \equiv \int_0^1 \Phi(c)\, dc \quad (\langle \Phi \rangle = \langle \Phi^* \rangle). \tag{2}$$

For simplicity, it is assumed here that the region of definition of the desired function c is specified by the interval [0, 1]; this may always be done by appropriate choice of the dimensionless function c. Determining the coefficients A and B requires the addition of one more relation, analogous to Eq. (2), which may be specified on the basis of various considerations. Below, the simplest and most natural choice of the constants A and B determining the function Φ^* in Eq. (1) and retaining the basic features of the initial function Φ is not used. Specifically, if the initial function $\Phi = \Phi(c)$ does not vanish in the interval [0, 1] and $\Phi = O(1)$, it will be approximated by a constant

$$\Phi^* = \langle \Phi \rangle \quad (A = \langle \Phi \rangle, \; B = 0), \tag{3}$$

whereas if Φ vanishes at some point $c_0 \in [0, 1]$, $\Phi(c_0) = 0$ then it is required that the approximating functions in Eqs. (1) and (2) also vanish at the same point $\Phi^*(c_0) = 0$; in particular, when $c_0 = 0$, this condition, together with Eq. (2), leads to the expression

$$\Phi^* = 2\langle \Phi \rangle c \quad (A = 0, \; B = 2\langle \Phi \rangle). \tag{4}$$

The accuracy of the given approximate method must naturally be estimated from the mean Sherwood number, which is also an integral mean.

The method here proposed is now illustrated for some specific problems. First, steady convective diffusion to a drop or a solid particle whan an arbitrary chemical reaction occurs

State Scientific-Research and Design Institute of the Nitrogen Industry and Products of Organic Synthesis, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 47, No. 2, pp. 205-215, August, 1984. Original article submitted January 11, 1983.

0022-0841/84/4702-0891$08.50

in the volume of the liquid is considered. In dimensionless variables, the corresponding boundary problem takes the form

$$\mathrm{Pe}\,(\mathbf{v}\nabla)\,c = \Delta c - kf\,(c) \quad (f\,(0) = 0,\ f'_c \geqslant 0,\ k \geqslant 0), \tag{5}$$

$$r = 1,\ c = 1;\ r \to \infty,\ c \to 0;$$

$$c = C/C_s,\ k = a^2 K_V F\,(C_s)/(DC_s),\ f\,(c) = F\,(C)/F\,(C_s). \tag{6}$$

Here r is a dimensionless radius (the radius referred to the particle radius a); $K_V F$ is the rate of chemical reaction.

The function f on the right-hand side of Eq. (5) vanishes when c = 0. This corresponds to the second case of the approximation in Eqs. (1) and (4) when $c_0 = 0$ and $\Phi = kf$, which leads to the following approximate boundary problem

$$\mathrm{Pe}\,(\mathbf{v}\nabla)\,c = \Delta c - 2k\,\langle f\rangle\,c;\ r = 1,\ c = 1;\ r \to \infty,\ c \to 0, \tag{7}$$

which corresponds to Eqs. (5) and (6).

The behavior of the mean Sherwood number corresponding to the solution of the accurate problem in Eqs. (5) and (6) and the approximate problem in Eq. (7) in some limiting cases will now be compared. First, note that, in the case of a first-order chemical reaction in the volume f(c) = c, Eq. (7) coincides with Eqs. (5) and (6) with any values of the parameters k and Pe, and therefore the accurate result is obtained, of course. When $k \to \infty$ (Pe = const), the mean Sherwood number corresponding to Eqs. (5) and (6) is determined by the formula [1]

$$\mathrm{Sh} = \left(2k\int_0^1 f\,(c)\,dc\right)^{1/2} \equiv (2k\,\langle f\rangle)^{1/2} \quad (k \to \infty,\ \mathrm{Pe} = \mathrm{const}). \tag{8}$$

It is evident that Eq. (8) does not change under the substitution $k \to 2k\langle f\rangle$ and $f(c) \to c$. This means that the use of the approximate Eq. (7) also leads in this case to the correct asymptotic result for the mean Sherwood number for any rate of chemical reaction in the liquid volume kf(c) (note that, despite the coincidence of the mean Sherwood numbers, the asymptotic solutions of the problems in Eqs. (5) and (6) and in Eq. (7) as $k \to \infty$ are different).

As $\mathrm{Pe} \to \infty$ (k = const), the solutions of the accurate problem in Eqs. (5) and (6) and the approximate solution in Eq. (2) coincide. Some discrepancy of the results corresponding to Eqs. (5) and (6) and Eq. (7) is observed only in the limiting case of small Peclet numbers $\mathrm{Pe} \to 0$ — i.e., when k = O(1).

It is clear from the foregoing that the best results from the use of the approximate Eq. (7) should be expected at large (and moderate) Peclet numbers over the whole range of variation of the parameter k for any function f = f(c).

If the mean Sherwood number, which is determined by the solution of the linear problem in Eq. (5) when f(c) = c is now denoted by J = J(k, Pe), the solution of the approximate problem in Eq. (7) corresponding to Eqs. (5) and (6) specifies the mean Sherwood number in the form $\mathrm{Sh} = J(2k\langle f\rangle, \mathrm{Pe})$. The dependence J = J(k, Pe) may be obtained, for example, by numerical solution of the problem; in addition, the results obtained by the method of asymptotic extrapolation may be used [2]. This allows an approximate expression to be written for a mean Sherwood number — corresponding to the problem in Eq. (5) — in the form

$$\mathrm{Sh}\,(k,\ \mathrm{Pe}) = 1 + [(\mathrm{Sh}\,(0,\ \mathrm{Pe}) - 1)^2 + 2k\,\langle f\rangle\,]^{1/2}. \tag{9}$$

The convective heat transfer of a drop or a solid particle in a liquid may now be considered with an arbitrary dependence of the thermal conductivity $\lambda_* = \lambda_*(T_*)$ on the temperature T_*, assuming that the temperature at the particle surface and far from it takes the constant values T_S and T_∞, respectively ($T_S \neq T_\infty$). For simplicity, it is assumed that inhomogeneity of the temperature does not influence the parameters of the flux. In dimensionless variables, the given nonlinear problem takes the form

$$\mathrm{Pe}\,(\mathbf{v}\nabla)\,T = \mathrm{div}\,(\lambda\nabla T);\ r = r_s,\ T = 0;\ r \to \infty,\ T \to 1;$$

$$T = \frac{T_s - T_*}{T_s - T_\infty},\ \lambda = \lambda\,(T) = \frac{\lambda_*\,(T_*)}{\lambda_*\,(T_s)},\ \mathrm{Pe} = \frac{aUc_p\rho}{\lambda_*\,(T_s)}\ (\lambda\,(0) = 1). \tag{10}$$

Here, as usual, it is assumed that the product of the specific heat and the density is constant: $c_{p\rho} = \text{const}$.

In problems of the type in Eq. (10), as a rule, $\lambda = O(1)$ when $T \in [0, 1]$; this corresponds to the first case of the approximation in Eq. (1) and (3) when $\Phi = \lambda$ and leads to the following approximate problem

$$\mathrm{Pe}\,(\mathbf{v}\nabla)\,T = \langle \lambda \rangle \Delta T;\ r = r_s,\ T = 0;\ r \to \infty,\ T \to 1. \tag{11}$$

In constrast to the nonlinear boundary problem in Eq. (10), the solution of the linear averaged problem in Eq. (11) has not yet been obtained in many specific cases. Note that, in the linear case $\lambda = 1$, the problems in Eqs. (10) and (11) coincide.

At small Peclet numbers $\mathrm{Pe} < O(1)$, the approximate problem in Eq. (11) gives the correct asymptotic result for the mean Nusselt number $\mathrm{Nu} = \mathrm{Nu}(\lambda, 0)$ for any form of the particles and an arbitrary dependence $\lambda = \lambda(T)$ [3]

$$\mathrm{Nu}\,(\lambda,\ 0) = \langle \lambda \rangle\ \mathrm{Nu}\,(1,\ 0)\left(\mathrm{Nu} = \frac{1}{4\pi}\int_S \lambda\,(T)\,\frac{\partial T}{\partial n}\,dT\right). \tag{12}$$

At large Peclet numbers, the nonlinear boundary problem in Eq. (10) has been scarcely investigated at all as yet. Only a few specific cases in which an accurate asymptotic solution of Eq. (10) as $\mathrm{Pe} \to \infty$ may be obtained will be considered here.

In the analysis for the velocity field of the liquid, consideration is limited to the principal term in the expansion of the current function (axisymmetric and plane cases are considered) close to the surface of the drop $\xi = 0$ ($n = 1$) or a solid particle ($n = 2$, 3)

$$\begin{gathered}\xi \to 0,\ \psi = \xi^n \sigma\,(\eta),\ \mathbf{v} = \{v_\xi,\ v_\eta,\ 0\}\quad (\sigma \geqslant 0),\\ v_\xi = -\sqrt{\frac{g^0_{\xi\xi}}{g^0}}\,\frac{\partial \psi}{\partial \eta},\quad v_\eta = \sqrt{\frac{g^0_{\eta\eta}}{g^0}}\,\frac{\partial \psi}{\partial \xi},\quad g^0 = g^0_{\xi\xi}\,g^0_{\eta\eta}\,g^0_{\varphi\varphi}.\end{gathered} \tag{13}$$

The coordinate η here is directed along the surface of the body ($\eta = 0$ corresponds to the inflow point) and ξ along the normal to the surface; it is assumed that the local value of ξ determines the distance from the point (ξ, η) to the surface of the body $(0, \eta)$, i.e., $g^0_{\xi\xi} = 1$); $g_{\xi\xi}$, $g_{\eta\eta}$, $g_{\varphi\varphi}$ are the components of the metric tensor; a superscript zero denotes that the corresponding value is taken at the body surface.

The approximation in Eq. (13) is valid in thermal problems for very viscous liquids when the thickness of the thermal boundary layer is considerably less than the thickness of the hydrodynamic boundary layer and is practically always valid in analogous diffusional problems.

Taking into account that, in the given boundary-layer approximation, $\mathrm{div}(\lambda \nabla T) \approx [\lambda T'_\xi]'_\xi$, and passing to the variable [4]

$$x = \mathrm{Pe}^{\nu}\psi^{1/n},\ \tau = \frac{1}{n}\int_0^\eta \sqrt{g^0}\,\sigma^{1/n}(\eta)\,d\eta,\ \nu = \frac{1}{n+1}, \tag{14}$$

the problem in Eq. (12) reduces to the following:

$$\begin{gathered}\frac{\partial T}{\partial \tau} - x^{1-n}\frac{\partial}{\partial x}\,\lambda\,(T)\,\frac{\partial T}{\partial x} = 0;\ \tau = 0,\ T = 1;\ x = 0,\\ T = 0;\ x \to \infty,\ T \to 1.\end{gathered} \tag{15}$$

The introduction of the new self-similar variable $z = x\tau^{-\nu}$ brings the problem in Eq. (15) to the form of an ordinary differential equation

$$\frac{d}{dz}\left[\lambda\,(T)\,\frac{dT}{dz}\right] + \nu z^n \frac{dT}{dz} = 0;\ z = 0,\ T = 0;\ z \to \infty,\ T \to 1. \tag{16}$$

When $n = 1$, Eq. (16) is often encountered in problems of nonlinear thermal conductivity [5] and filtration [6] and, for any dependences $\lambda(T)$, its solution has already been obtained by analytical or numerical methods.

To compare the accurate and averaged ($\lambda \to \langle\lambda\rangle$) solutions of Eq. (16) when $n = 1$, consider the linear temperature dependence of the thermal conductivity which is often employed

$$n = 1,\ \lambda(T) = 1 + \varkappa T,\ \langle \lambda \rangle = 1 + \frac{1}{2}\varkappa. \tag{17}$$

When $\varkappa = 0$, the solution of the accurate and averaged problems in Eqs. (16) and (17) coincide. When $\varkappa \neq 0$, the solution of the accurate problem may be found in [7], and the solution of the averaged problem takes the form $T = \mathrm{erf}(z/\sqrt{\langle\lambda\rangle})$. The maximum error introduced by the averaging procedure will correspond to the transition in Eq. (17) to the limit as $\varkappa \to \infty$. In this case, the first term (unity) on the right-hand side of the expressions for λ and $\langle\lambda\rangle$ in Eq. (17) may be neglected. The accurate solution of the problem in Eq. (16) when $\lambda(T) = \varkappa T$ was obtained in [6] and leads to the following local flux at the drop surface

$$J = \left[\lambda(T)\frac{dT}{dz}\right]_{z=0} = 0{,}664\sqrt{\varkappa};\quad J^* = 2\sqrt{\frac{\langle\lambda\rangle}{\pi}} \approx 0{,}798\sqrt{\varkappa}. \tag{18}$$

The second expression here corresponds to the local flux for the averaged Eq. (16) as $\varkappa \to \infty$. Comparison of the expressions in Eq. (18) shows that the maximum error introduced by the averaging procedure is 20% in this case. Note, however, that, in the limiting case as $\varkappa \to \infty$, the basic condition of applicability of the averaging procedure is not satisfied — that is, the condition that $\lambda(T) = O(1)$ when $T \in [0, 1]$. In real situations, when $\lambda = O(1)$, the error in using the approximate equation may be considerably less than 20%.

It follows from the results of [7] that an analogous error is also observed in averaging the nonsteady boundary problems corresponding to Eq. (10). In particular, in approximating the flow field in the vicinity of the drop surface, by the principal term of the expansion of the current function in the form $\xi \to 0$, $\psi = \xi\sigma(t, \eta)$ — cf. Eq. (13) — there is a variable change [7] which allows the nonsteady boundary problem in Eq. (10) to be reduced to an ordinary differential equation, Eq. (16) with $n = 1$; here $z = z(t, \psi, \eta)$ has already been determined in the course of solving the problem. Then, in the case of a linear temperature dependence of the thermal diffusivity in Eq. (17), the maximum error in using the averaging procedure is also no more than 20%.

The next step is to write the problem in Eq. (10) in which the averaging procedure for the thermal diffusivity in Eq. (13) is asymptotically accurate for large Peclet numbers (for the mean Sherwood number). Specifically, it is shown that Eq. (13) gives a correct asymptotic result: 1) in doubly connected regions with completely closed streamlines (Fig. 1a); 2) in all external problems in which a region with completely closed streamlines is adjacent to the surface of the body (Fig. 1b) (this case corresponds, for example, to the motion of a freely suspended spherical particle of cylinder in a linear shear flux [8, 9]).

Consider the first case first. For analysis, in addition to the spherical coordinate system r, θ, φ, it is expedient to introduce an orthogonal coordinate system ψ, χ, φ associated with the streamlines; the fixed curves $\chi = \text{const}$ are orthogonal to the streamlines $\psi = \text{const}$; the dependence $\chi = \chi(r, \theta, \varphi)$ is determined by solving the equation $(\nabla\psi\nabla\chi) = 0$, only the plane or axisymmetric case, i.e., $\partial/\partial\varphi = 0$ being considered here. In this coordinate system, Eq. (10) takes the form

$$\frac{\mathrm{Pe}}{\sqrt{g}}\frac{\partial T}{\partial \chi} = \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial\psi}\left[\frac{\sqrt{g}}{g_{\psi\psi}}\lambda(T)\frac{\partial T}{\partial\psi}\right] + \frac{\partial}{\partial\chi}\left[\frac{\sqrt{g}}{g_{\chi\chi}}\lambda(T)\frac{\partial T}{\partial\chi}\right]\right\};$$
$$\mathbf{v} = (0,\ v_\chi,\ 0),\ v_\chi = (g_{\chi\chi}/g)^{1/2},\ g = g_{\psi\psi}g_{\chi\chi}g_{\varphi\varphi} \tag{19}$$
$$(\psi_1 \leqslant \psi \leqslant \psi_2,\ 0 \leqslant \chi \leqslant \chi_0).$$

In the plane case $g_{\varphi\varphi} = 1$ and in the axisymmetric case $g_{\varphi\varphi} = r^2\sin^2\theta$.

To Eq. (19) must be added boundary conditions on the limiting streamlines

$$\psi = \psi_1,\ T = 0;\quad \psi = \psi_2,\ T = 1. \tag{20}$$

The dimensionless temperature T in Eqs. (19) and (20) is introduced analogously as in Eq. (10), with the corresponding substitution $T_S \to T_{S1}$, $T_\infty \to T_{S2}$ (T_{S1} and T_{S2} are the temperatures at the surfaces ψ_1 and ψ_2). The solution of Eqs. (19) and (20) as $\mathrm{Pe} \to \infty$ is sought in the form of a regular asymptotic expansion in inverse powers of the Peclet number

$$T = T_0 + \mathrm{Pe}^{-1}T_1 + \ldots,\ T_0/T_1 = O(1). \tag{21}$$

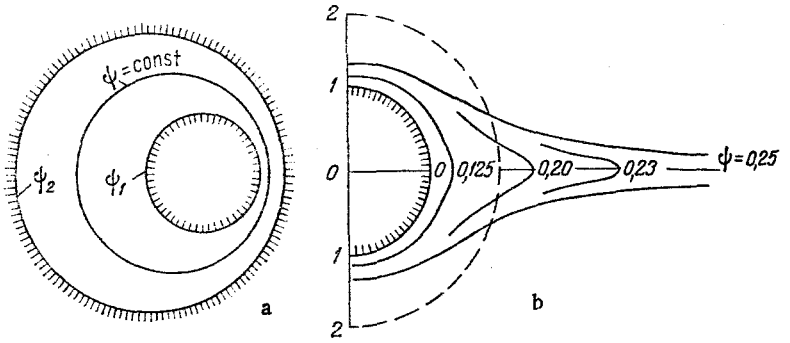

Fig. 1. Streamlines in the gap between two (rotating) cylinders (a) and in the vicinity of a circular cylinder freely rotating in a simple shear flow (b).

Substitution of Eq. (21) into Eqs. (19) and (20) gives, after isolating the terms with identical powers of the Peclet number for the principal terms of the expansion, the result

$$\frac{\partial T_0}{\partial \chi} = 0;\ \psi = \psi_1,\ T_0 = 0;\ \psi = \psi_2,\ T_0 = 1; \tag{22}$$

$$\frac{\partial T_1}{\partial \chi} = \frac{\partial}{\partial \psi}\left[\frac{\sqrt{g}}{g_{\psi\psi}}\lambda(T_0)\frac{\partial T_0}{\partial \psi}\right] + \frac{\partial}{\partial \chi}\left[\frac{\sqrt{g}}{g_{\chi\chi}}\lambda(T_0)\frac{\partial T_0}{\partial \chi}\right]; \tag{23}$$

$$\psi = \psi_1,\ T_1 = 0;\ \psi = \psi_2,\ T_1 = 0.$$

It follows from Eq. (22) that the zero term of the expansion depends solely on the current function (it is assumed that $\partial/\partial\varphi = 0$)

$$T_0 = T_0(\psi). \tag{24}$$

The boundary conditions in Eq. (22) are then found to be insufficient for the determination of the temperature T_0. The necessary additional information on the zero term of the expansions in Eqs. (21) and (24) is obtained using a procedure often encountered in applying the expanded-coordinate method and the multiple-scale method with successive elimination of the secular (increasing) terms [10, 11].

Taking into account that the function T is periodic in the coordinate χ (with period χ_0)

$$T(\psi, \chi) = T(\psi, \chi + \chi_0)\ (T_i(\psi, \chi) = T_i(\psi, \chi + \chi_0);\ i = 0, 1, \ldots), \tag{25}$$

the equation is integrated for the next term of the expansion of T_1 in Eq. (23) with respect to χ: $0 \leqslant \chi \leqslant \chi_0$. As a result, an ordinary differential equation is obtained for T_0.

$$\frac{d}{d\psi}\left[\Gamma(\psi)\lambda(T_0)\frac{dT_0}{d\psi}\right] = 0;\ \psi = \psi_1,\ T_0 = 0;\ \psi = \psi_2,\ T_0 = 1;$$

$$\Gamma(\psi) = \oint \frac{\sqrt{g}}{g_{\chi\chi}}d\chi\ \left(\oint d\chi \equiv \int_0^{\chi_0} d\chi\right). \tag{26}$$

The solution of Eq. (26) (written in implicit form) is defined by the formulas

$$\int_0^{T_0}\lambda(T)\,dT = \langle\lambda\rangle\frac{\Lambda(\psi, \psi_1)}{\Lambda(\psi_2, \psi_1)},\ \Lambda(\psi, \psi_1) = \int_{\psi_1}^{\psi}\frac{d\psi}{\Gamma(\psi)}, \tag{27}$$

and the corresponding mean Nusselt number $\mathrm{Nu}(\lambda, \mathrm{Pe})$ takes the form

$$\mathrm{Nu}(\lambda, \infty) = \langle\lambda\rangle\,\mathrm{Nu}(1, \infty),\ \mathrm{Nu}(1, \infty) = \Lambda^{-1}(\psi_2, \psi_1). \tag{28}$$

It is not difficult to show, analogously, that the asymptotic solution of the "averaged" problem in Eq. (11) at large Peclet numbers is specified by Eq. (27) at $\lambda = \langle\lambda\rangle = 1$. The corresponding mean Nusselt number is determined by integrating the local diffusional flux $j = \langle\lambda\rangle(\sqrt{g}g_{\psi\psi}^{-1}dT_0/d\psi)_{\psi=\psi_1}$ with respect to the period χ_0, and leads to the same expression, Eq. (28) — i.e., in the given case, the "averaged" Eq. (11) leads to an accurate asymptotic result for the mean Sherwood number. Note that Eqs. (27) and (28) generalize the results of [9, 12] to the nonlinear case.

In the second case (Fig. 1b), boundary conditions at the surface of the body and at infinity are imposed. It may readily be shown that at large Peclet numbers the temperature distribution in the flow is also described here by the ordinary differential equation in Eq. (26), where $\psi = \psi_1$ corresponds to the surface of the body and $\psi = \psi_2$ to the limiting streamline, which separates the region with closed streamlines from the region with open streamlines; the boundary condition at infinity is "transferred" to this limiting streamline, i.e., the concentration in the region with open current lines is constant, and equal to the unperturbed value at infinity. The given case is realized, for example, with linear shear flow around an unfixed circular cylinder, when the liquid velocity field at infinity in the Cartesian coordinate system x_1, x_2, x_3 may be written in the form $\mathbf{v} = G\{x_2, 0, 0\}$, where G is the shear coefficient. In Fig. 1b, the streamlines ψ = const are shown, where

$$\psi = \frac{1}{4}(r^2 - 1) - \frac{1}{4}(r^2 - 2 + r^{-2})\cos 2\theta, \quad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r}, \tag{29}$$

which corresponds to such shear flow around a circular cylinder (which rotates at constant angular velocity) [8, 9]; in writing the dimensionless current function in Eq. (29), the quantity $U = aG$ (a is the cylinder radius) is taken as the characteristic velocity scale. The limiting streamline dividing the regions with closed and open streamlines is determined by the relation $\psi = 1/4$ in Eq. (26).

The asymptotic solution of the problem in Eq. (10) as $Pe \to \infty$ in the second case (Fig. 1b) is also specified by Eq. (27). The mean Nusselt number over the surface of the body corresponding to the solution of the accurate problem in Eq. (10) and the averaged problem in Eq. (11) coincide here, and are determined by Eq. (28).

Note that, in the particular case of the heat transfer of an unfixed circular cylinder in linear shear flow in Eq. (29) at large Peclet numbers, with any law of variation in the thermal conductivity $\lambda = \lambda(T)$ as a function of the temperature, the mean Nusselt number must be determined using Eq. (28), where $Nu(1, \infty) = 2.87$ [9]; the dimensionless integral flux is $2\pi Nu(1, \infty)$. In the case of the heat transfer of a sphere freely suspended in a shear flow (a three-dimensional flow field), $Nu(1, \infty) \approx 4.5$ must be assumed in Eq. (28) [13].

It is clear from the given comparisons with accurate asymptotic results that the "averaged" Eq. (11) may be successfully used for the approximate determination of the mean Sherwood number.

Earlier in [3], asymptotic analysis of the nonlinear boundary problem in Eq. (10) at small Peclet numbers is performed in the case of translational and arbitrary shear flow for any form of drops and particles. For the mean Nusselt number $Nu(\lambda, Pe)$, the following formula is obtained

$$Nu(\lambda, Pe) = \langle \lambda \rangle Nu(1, Pe), \quad \langle \lambda \rangle = \int_0^1 \lambda(T)\, dT, \tag{30}$$

which allows the first three terms (in the case of a translational flow) and the first two terms (in the case of shear flow) in the asymptotic expansion with respect to the small Peclet number to be determined. Here $Nu(1, Pe)$ is the mean Nusselt number corresponding to the linear problem in Eq. (10) with a constant thermal conductivity $\lambda = 1$, which was obtained in [14, 15], respectively, for translational and shear flow around a particle.

It follows from a comparison of Eqs. (28) and (30) that Eq. (30) may successfully be used for the aprpoximate determination of the mean Nusselt number over the whole range of Peclet numbers, with any temperature dependence of the thermal conductivity, for a broad class of flows (in any case, with shear flow around freely suspended particles).

2. Method of Boundary-Condition Averaging

In nonlinear boundary problems described by linear partial differential equations with nonlinear boundary conditions at the surface of the body (nonlinear boundary conditions of the third kind), a simple approximate method based on satiafying the boundary conditions at the particle surface "on average" is often useful for approximate determination of the integral thermal and diffusional fluxes at the particle surface [16]. The basic idea of the method is illustrated in the following example, which is of independent interest.

Consider convective diffusion to a reacting solid or liquid sphere immersed in a laminar liquid flow when an isothermal chemical reaction occurs at the surface of the body at a rate

depends in an arbitrary manner on the temperature and the concentration. It is assumed that the particle is not heat-conducting, and that the reacting components are present in concentrations sufficiently small for the presence of a surface chemical reaction to have no influence on the parameters of the flow and the particle. No account is also taken either of the thermal and baric diffusion, and so on.

The dimensionless equations of convective diffusion and thermal conductivity and the boundary conditions expressing the homogeneity of the temperature and concentration far from the particle, together with the "reaction law" and the heat balance at its surface, take the form

$$\mathrm{Pe}_m (\mathbf{v}\nabla) c_m = \Delta c_m,\ m = 1, 2, \ldots, M, \tag{31}$$

$$\mathrm{Pe}_0 (\mathbf{v}\nabla) T = \Delta T, \tag{32}$$

$$r \to \infty,\ c_m \to 0,\ T \to 0, \tag{33}$$

$$r = 1,\ -\frac{\partial c_m}{\partial r} = f_m (T, c_1, \ldots, c_M), \tag{34}$$

$$r = 1,\ \frac{\partial T}{\partial r} = \sum_{m=1}^{M} h_m \frac{\partial c_m}{\partial r}, \tag{35}$$

$$C_m = C_{m\infty}(1 - c_m),\ T_* = T_\infty (1 - T),$$

$$\mathrm{Pe}_m = aUD_m^{-1},\ \mathrm{Pe}_0 = aU\sigma^{-1},$$

$$h_m = C_{m\infty} H_m D_m (\lambda T_\infty)^{-1},\ f_m (T, c_1, \ldots, c_M) = a (D_m C_{m\infty})^{-1} F_m (T_*, C_1, \ldots, C_M).$$

It is impossible to obtain an effective solution of the problem in Eqs. (31)-(35) even in the simplest linear case when M = 1 (with arbitrary Peclet numbers $0 \leqslant \mathrm{Pe} < \infty$). Therefore, the approximate solution in Eqs. (31)-(35) is constructed as follows. An (M + 1)-parameter family of functions c_m and T satisfying Eqs. (31) and (32) and boundary conditions at infinity are taken

$$c_m = A_m u_m,\ T = Bw,\ u_m = \Xi (\mathbf{r}, \mathrm{Pe}_m),\ w = \Xi (\mathbf{r}, \mathrm{Pe}_0),$$
$$A_m = \mathrm{const},\ B = \mathrm{const}\ (m = 1, \ldots, M). \tag{36}$$

Here the function $\Xi = \Xi(\mathbf{r}, \mathrm{Pe})$ is a solution of the following linear boundary problem

$$\mathrm{Pe}\,(\mathbf{v}\nabla)\Xi = \Delta\Xi;\ r = 1,\ \Xi = 1;\ r \to \infty,\ \Xi \to 0, \tag{37}$$

and the arbitrary constants A_m and B will be determined below in the course of solving the problem.

It may readily be established, by direct verification, that the function $c_m = A_m u_m$ and $T = Bw$ in Eqs. (36) and (37) with any constants A_m and B satisfy Eqs. (31) and (32) and the boundary problems at infinity in Eq. (33). The unknown constants A_m and B are determined from the condition that the nonlinear boundary conditions in Eqs. (34) and (35) are equal on average over the particle surface

$$r = 1,\ -\int_S \frac{\partial c}{\partial r} dS = \int_S f_m dS;\ \int_S \frac{\partial T}{\partial r} dS = \int_S \left(\sum_{m=1}^{M} h_m \frac{\partial c_m}{\partial r} \right) dS. \tag{38}$$

Substituting Eqs. (36) and (37) into these formulas, and reversing the order of summation and integration on the right-hand side of the last relation in Eq. (38), the following transcendental (algebraic) equations are obtained for determining A_m and B

$$\mathrm{Sh}_m = f_m (B, A_1, \ldots, A_M);\ \mathrm{Nu} = \sum_{m=1}^{M} h_m \mathrm{Sh}_m;$$

$$\mathrm{Sh}_m = -\frac{1}{4\pi} \int_S \frac{\partial c_m}{\partial r} dS;\ \mathrm{Nu} = -\frac{1}{4\pi} \int_S \frac{\partial T}{\partial r} dS \quad (m = 1, \ldots, M). \tag{39}$$

It is also taken into account that, in view of Eqs. (36) and (37), there is a simple relation between the mean Sherwood and Nusselt numbers and the parameters A_m and B

$$\mathrm{Sh}_m = A_m \mathrm{Sh}_{m\infty};\ \mathrm{Nu} = B\,\mathrm{Nu}_\infty;$$

$$\mathrm{Sh}_{m\infty} = -\frac{1}{4\pi}\int_S \frac{\partial u_m}{\partial r}\,dS;\ \mathrm{Nu}_\infty = -\frac{1}{4\pi}\int_S \frac{\partial w}{\partial r}\,dS, \tag{40}$$

where $\mathrm{Sh}_{m\infty}$ and Nu_∞ correspond to purely diffusional (thermal) reaction conditions at the surface of the sphere in Eq. (37).

Excluding the parameters A_m and B from Eq. (40), and substituting them into Eq. (39), the following system of transcendental equations is obtained for determining the mean Sherwood and Nusselt numbers

$$\mathrm{Sh}_m = f_m\left(\frac{\mathrm{Nu}}{\mathrm{Nu}_\infty}, \frac{\mathrm{Sh}_1}{\mathrm{Sh}_{1\infty}}, \ldots, \frac{\mathrm{Sh}_M}{\mathrm{Sh}_{M\infty}}\right);\ m = 1, \ldots, M;$$

$$\mathrm{Nu} = \sum_{m=1}^{M} h_m \mathrm{Sh}_m. \tag{41}$$

Note that, in order to determine the auxiliary Sherwood $\mathrm{Sh}_{m\infty}$ and Nusselt Nu_∞ numbers appearing in Eq. (41), it is sufficient to know the solution of only one of the linear relations in Eq. (31) over the whole range of Peclet numbers $0 \leqslant \mathrm{Pe} < \infty$. As already noted, there are at present sufficiently many solutions of the problem in Eq. (37) obtained for different conditions of flow around a drop or a particle by numerical, analytical, or approximate methods.

In the isothermal case, when M = 1, the suitability of the approximate Eq. (41) in various specific situations was checked in [1, 17], where it was shown that the maximum deviation of the roots of the approximate Eq. (41) from the accurate mean Sherwood number is observed at large Peclet numbers ($\mathrm{Pe} \to \infty$) and does not exceed 7-9% as a rule.

Note that the approximate system of transcendental equations in Eq. (41) is asymptotically accurate at small Peclet numbers $\mathrm{Pe} \to 0$, $\mathrm{Pe}_m = \mathrm{Pe}Q_m$, $Q_m = O(1)$, and allows the first three terms (in the case of a translational flow [18]) and the first four terms (in the case of arbitrary linear shear flow [19]) of the corresponding asymptotic expansion in small Peclet numbers to be obtained for the mean Sherwood and Nusselt numbers.

NOTATION

C, concentration in flow; C_S, concentration at particle surface; C_∞, unperturbed concentration at infinity; C_m, concentration of the m-th component; $C_{m\infty}$, concentration of the m-th component at infinity; $c = C/C_S$, dimensionless concentration; $c_m = (C_{m\infty} - C_m)/C_{m\infty}$; D, D_m, diffusion coefficients; F_m, rate of surface reaction of the m-th component; $f_m = a(D_m C_{m\infty})^{-1} F_m$, dimensionless reaction rate of the m-th component; G, shear coefficient; $g_{\xi\xi}$, $g_{\eta\eta}$, $g_{\varphi\varphi}$, components of the metric tensor; H_m, heat of reaction of the m-th component; $h_m = H_m C_{m\infty} D_m (\lambda T_\infty)^{-1}$; K_V, rate constant of chemical reaction in the liquid volume; k, dimensionless reaction rate constant; M, number of reagents involved in reaction; Nu, mean Nusselt number; Pe, Peclet number; Pe_0, Pe_m, thermal and diffusional Peclet numbers; r, θ, φ, spherical coordinate system associated with the particle; Sh, Sh_m, mean Sherwood number; T*, temperature in flow; T_∞, temperature at infinity; T_S, temperature at particle surface; T, dimensionless temperature; U, characteristic velocity of flow; x_1, x_2, x_3, Cartesian coordinate system; λ, thermal conductivity; ξ, η, φ, local orthogonal curvilinear coordinate system associated with particle surface; ρ, liquid density; σ, thermal diffusivity; ψ, current function; ψ, χ, φ, curvilinear coordinate system associated with the current function; $r = r_S(\theta, \varphi)$, equation of particle surface (for the sake of brevity, the arguments θ and φ of the function r_S are omitted in the text).

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